General formula for stability testing of fractional-delay systems

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Abstract—**An easy-to-use and effective formula for stability testing of a system with fractional-delay characteristic equation in the general form of** $\Delta(s) = P_0(s) + \sum_{i=1}^{N} P_i(s) \exp(-\zeta_i s^{\beta_i}) = 0$, **where** $P_i(s)$ $(i = 0, \ldots, N)$ **are the so-called fractional-order polynomials and** ζ_i **and** β_i **are positive real constants, is proposed in this paper. The proposed formula determines the number of unstable roots of the characteristic equation (i.e., those located in the right half-plane of the first Riemann sheet) by applying Rouche's theorem. Numerical simulations are also presented to confirm the efficiency of the proposed formula.**

I. INTRODUCTION

In some of the recently-developed control problems we need to check the stability of a system with the so-called *fractional-delay* characteristic equation in the general form of

$$\Delta(s) = P_0(s) + \sum_{i=1}^{N} P_i(s) \exp(-\zeta_i s^{\beta_i}) = 0, \qquad (1)$$

where ζ_i and β_i are positive real constants and $P_i(s)$ $(i = 1, \ldots, N)$ are fractional-order polynomials in the form of

$$P_i(s) = \sum_{k=1}^{M_i} a_{ik} s^{\alpha_{ik}}, \qquad (2)$$

where a_{ik} and α_{ik} are real and positive real constants, respectively, and

$$P_0(s) = s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_1 s^{\alpha_1} + a_0, \qquad (3)$$

where, without any loss of generality, it is assumed that the powers of s in (3) satisfy the following relations:

$$\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 0. \qquad (4)$$

As an example of a system with fractional-delay characteristic equation, consider a classical unity-feedback system in which a process with transfer function

$$G(s) = \frac{K}{1 + sT} e^{-sL}, \qquad (5)$$

is controlled with the so-called PI$^\lambda$D$^\mu$ controller with transfer function [1]:

$$C(s) = K_p \left(1 + \frac{1}{T_i s^\lambda} + T_d s^\mu \right), \qquad (6)$$

where K_P, T_i, T_d, λ, and μ are unknown parameters of the controller to be determined. It can be easily verified that the characteristic equation of this system is as the following

$$\Delta(s) = T_i s^\lambda (1 + sT) + K_p K(T_i s^\lambda + 1 + T_i T_d s^{\lambda + \mu}) e^{-sL} = 0, \qquad (7)$$

which can be considered as a special case of (1) with $N = 1$,

$$P_0(s) = T_i s^\lambda (1 + sT), \qquad (8)$$

$$P_1(s) = K_p K(T_i s^\lambda + 1 + T_i T_d s^{\lambda + \mu}), \qquad (9)$$

$\zeta_1 = L$, and $\beta_1 = 1$. If in this example one tries to find the optimal values of K_P, T_i, T_d, λ, and μ by means of meta-heuristic optimization algorithms such that a certain cost function (e.g., ISE performance index corresponding to the tracking of unit step command) is minimized, he/she will need a method to check the feasibility of the solutions generated by the meta-heuristic optimization algorithm from the stability point of view. It should be noted that since in such optimization problems the cost function is usually expressed in the frequency domain (by applying Parseval's theorem), the resulted optimal controller may destabilize the feedback system [2].

As a more complicated example, consider the problem of designing an optimal PI$^\lambda$D$^\mu$ controller for a process whose transfer function consists of fractional powers of s possibly in combination with exponentials of fractional powers of s. For example, the transfer functions:

$$G(s) = \frac{\cosh(x_0 \sqrt{s})}{\sqrt{s} \sinh(\sqrt{s})}, \quad 0 < x_0 < 1, \qquad (10)$$

and

$$G(s) = \frac{\sinh(x_0 \sqrt{s})}{\sinh(\sqrt{s})}, \quad 0 < x_0 < 1, \qquad (11)$$

appear in boundary control of one-dimensional heat equation with Neumann and Dirichlet boundary conditions [3]. Other examples of this type can be found in [3]-[5]. Moreover, in some applications in order to arrive at more accurate models, the process is modelled with a fractional-order transfer function. For instance, Podlubny [6] showed that the fractional-order transfer function:

$$G(s) = \frac{1}{0.7943 s^{2.5708} + 5.2385 s^{0.8372} + 1.5560}, \qquad (12)$$

can better model a heating furnace compared to classical integer-order transfer functions. Clearly, in dealing with complicated transfer functions such as those given in (10)-(12) we need more powerful tools to determine the stability of the corresponding closed-loop system.

Stability analysis of the feedback system when such complicated transfer functions exist in the loop is a challenging task. Even the stability analysis of a feedback system which consists of both $PI^\lambda D^\mu$ controller and a process with dead-time is not straightforward. So far, many researchers have tried to develop analytical or numerical methods for stability testing of systems with fractional-delay characteristic equations (see [7] for a detailed review of some important works in relation to the stability testing of fractional-delay systems). Probably, the most famous analytical method for stability testing of fractional-order systems (as a special case of fractional-delay systems) is the *sector stability test* of Matignon [8], which was already reported in the work of Ikeda and Takahashi [9]. Application of this method is limited to the case where the sigma term does not exist in (1) and $P_0(s)$ is of *commensurate order*. Few numerical algorithms for stability testing of (1) can also be found in the literature (see, for example, [10] and [7] and the references therein for more information on this subject). As far as we know, all of these methods suffer from the limitation that can be applied only to a certain class of fractional-delay systems [10], or the results are of probabilistic nature [7].

The aim of this paper is to propose a formula for determining the number of unstable roots of (1). The proposed formula is actually a generalization of the method already proposed by author in [10]. However, the formula developed in this paper has the advantage of being much simpler compared to the one presented in [10], and moreover, it can be easily applied to a more general form of fractional-delay characteristic equations.

The rest of this paper is organized as follows. The proposed formula for stability testing of fractional-delay characteristic equations is presented in Section II. Four numerical examples are studied in Section III, and finally, Section IV concludes the paper.

II. PROPOSED FORMULA FOR STABILITY TESTING OF FRACTIONAL-DELAY CHARACTERISTIC EQUATIONS

The first step in dealing with multi-valued complex functions (such as the one presented in (1)) is to construct the domain of definition of the function appropriately. The domain of definition of the characteristic function given in (1) is, in general, in the form of a Riemann surface with infinite number of Riemann sheets, where the origin is a branch point and the branch cut is considered (arbitrarily) at $\mathbb{R}^-$. Equation $\Delta(s) = 0$ as defined in (1) has, in general, infinite number of roots which are distributed on this Riemann surface. As a well-known fact, a system with characteristic equation (1) is stable if and only if it does not have any roots in the right half-plane of the first Riemann sheet [7], [11]. Hence, stability analysis of a system with characteristic equation (1) is equivalent to

investigation for the roots of $\Delta(s) = 0$ in the right half-plane of the first Riemann sheet. In the following we will use Rouche's theorem for this purpose.

First, let us briefly review the Rouche's theorem. Consider the complex function $f : \mathbb{C} \to \mathbb{C}$ which has zeros of orders $m_1, \ldots, m_k$ respectively at $z_1, \ldots, z_k$ and does not have any poles. This function can be written as

$$f(s) = g(s)(s - z_1)^{m_1} \times (s - z_2)^{m_2} \times \cdots \times (s - z_k)^{m_k}, \quad (13)$$

where $g(s)$ has neither pole nor zero. Taking the natural logarithm from both sides of (13) leads to

$$\ln f(s) = \ln g(s) + m_1 \ln(s - z_1) + m_2 \ln(s - z_2) + \ldots \\ + m_k \ln(s - z_k). \quad (14)$$

Derivation with respect to s from both sides of (14) yields

$$\frac{f'(s)}{f(s)} = \frac{g'(s)}{g(s)} + \frac{m_1}{s - z_1} + \frac{m_2}{s - z_2} + \ldots + \frac{m_k}{s - z_k}. \quad (15)$$

Now let γ be a simple, closed, counterclockwise contour such that $f(s)$ has no zeros (or singularities like branch point and branch cut in dealing with multi-valued functions) on it. Then it is concluded from the Residue theorem that

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(s)}{f(s)} \mathrm{d}s = M, \quad (16)$$

where M is equal to the total number of the roots of $f(s) = 0$ inside γ. Clearly, if the contour γ is considered such that all zeros of $f(s)$ lie inside it then we have $M = \sum_{j=1}^{k} m_j$. Equation (16) can be used to calculate the number of zeros of the given function $f(s)$ inside the desired contour γ (which, of course, should have the above-mentioned properties). For this purpose, we can simply use a numerical integration technique to evaluate the integral in the right hand side of (16) for the given contour γ and function f.

According to the above discussions, by setting $f(s)$ equal to $\Delta(s)$ and γ equal to the border of the region of instability (which is equal to the closed right half-plane of the first Riemann sheet) the value obtained for M from (16) will be equal to the number of unstable roots of the characteristic equation. In the following, we consider the contour γ as shown in Fig. 1 and $f(s) = \Delta(s)$ (where $\Delta(s)$ is defined in (1)) and then simplify the integral in the left hand side of (16) to arrive at a more effective formula for stability testing of the fractional-delay system under consideration (clearly, the system is stable if and only if $M = 0$). Note that the very small semicircle in Fig. 1 is used to avoid the branch-point located at the origin.

According to (16) and Fig. 1 we can write

$$M = \frac{1}{2\pi i} \oint_\gamma \frac{\Delta'(s)}{\Delta(s)} \mathrm{d}s = \frac{1}{2\pi i} \left(\int_{c_1 + c_3} + \int_{c_2} + \int_{c_4} \right). \quad (17)$$
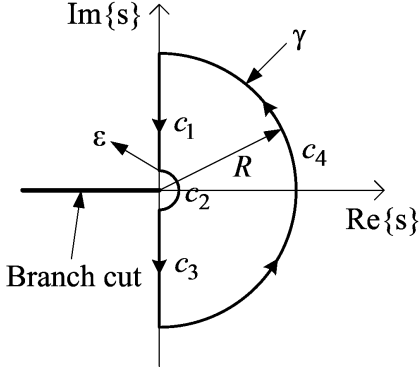
Fig. 1. The contour γ considered on the first Riemann sheet for stability testing of the fractional-delay system under consideration. A system with characteristic equation $\Delta(s) = 0$ (as defined in (1)) is stable if and only if it does not have any roots inside γ.

In (17), the integral $\int_{c_1+c_3}$ is calculated as

$$\int_{c_1+c_3} = \int_{\infty}^{\varepsilon} \frac{\Delta'(i\omega)}{\Delta(i\omega)} i\mathrm{d}\omega + \int_{-\varepsilon}^{-\infty} \frac{\Delta'(i\omega)}{\Delta(i\omega)} i\mathrm{d}\omega \qquad (18)$$

$$= -\int_{\varepsilon}^{\infty} \frac{\Delta'(i\omega)}{\Delta(i\omega)} i\mathrm{d}\omega + \int_{\varepsilon}^{\infty} \frac{\Delta'(-i\omega)}{\Delta(-i\omega)} (-i)\mathrm{d}\omega \quad (19)$$

$$= -i\int_{\varepsilon}^{\infty} \frac{\Delta'(i\omega)}{\Delta(i\omega)} \mathrm{d}\omega - i\int_{\varepsilon}^{\infty} \left(\frac{\Delta'(i\omega)}{\Delta(i\omega)}\right)^* \mathrm{d}\omega, \quad (20)$$

which yields

$$\int_{c_1+c_3} = -2i\int_{\varepsilon}^{\infty} \mathrm{Re}\left\{\frac{\Delta'(i\omega)}{\Delta(i\omega)}\right\} \mathrm{d}\omega. \qquad (21)$$

The integral $\int_{c_2}$ in (17) is calculated as

$$\int_{c_2} = \lim_{\varepsilon \to 0} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\Delta'(\varepsilon e^{i\theta})}{\Delta(\varepsilon e^{i\theta})} \varepsilon i e^{i\theta} \mathrm{d}\theta \qquad (22)$$

$$= \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \lim_{\varepsilon \to 0} \left\{\varepsilon \frac{\Delta'(\varepsilon e^{i\theta})}{\Delta(\varepsilon e^{i\theta})}\right\} i e^{i\theta} \mathrm{d}\theta. \qquad (23)$$

In the above equation $\lim_{\varepsilon \to 0} \Delta(\varepsilon e^{i\theta})$ is equal to a nonzero constant (else, the characteristic function has a strong singularity at the origin and the corresponding system is unstable) and $\Delta'(\varepsilon e^{i\theta}) \sim K\varepsilon^{\eta}$ as $\varepsilon \to 0$, where K and $\eta > -1$ are two constants. Hence, $\int_{c_2}$ tends to zero as $\varepsilon \to 0$. Finally, $\int_{c_4}$ in (17) is calculated as

$$\int_{c_4} = \lim_{R \to \infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\Delta'(Re^{i\theta})}{\Delta(Re^{i\theta})} Ri e^{i\theta} \mathrm{d}\theta \qquad (24)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lim_{R \to \infty} \left\{\frac{\Delta'(Re^{i\theta})}{\Delta(Re^{i\theta})} R\right\} i e^{i\theta} \mathrm{d}\theta \qquad (25)$$

$$= i\alpha_n \pi. \qquad (26)$$

(See (1) and (3) for the definition of α_n.) Substitution of (21) and (26) in (17) and considering the fact that $\int_{c_2} = 0$ results in

$$M = \frac{\alpha_n}{2} - \frac{1}{\pi} \int_{\varepsilon=0^+}^{\infty} \mathrm{Re}\left\{\frac{\Delta'(i\omega)}{\Delta(i\omega)}\right\} \mathrm{d}\omega, \qquad (27)$$

where M is equal to the number of unstable poles of a system with characteristic equation $\Delta(s) = 0$ as defined in (1).

Equation (27) is the main result of this paper. It should be noted that the value of ε in (27) cannot, in general, be considered exactly equal to zero. That is because of the fact that the numerical integration technique used to evaluate the integral in (27) performs this task by evaluating the integrand at different points of the ω axis. Hence, the numerical integration algorithm may halt if the integrand becomes singular at the origin (which is the case if, for example, $0 < \alpha_1 < 1$ in (3)). In practice, in order to determine the number of unstable poles of the given fractional-delay transfer function we can consider the lower and upper bound of the integral in (27) equal to sufficiently small and big positive numbers, respectively. The MATLAB function quadl (as well as quadgk) can be used to evaluate the integral in (27). Some numerical examples will be presented in the next section.

III. NUMERICAL EXAMPLES

In the following we study the application of (27) for stability testing of some fractional-delay systems. In each case, the impulse response of the corresponding system is also plotted to verify the correctness of the result. The method used in this paper to calculate the impulse response of the given fractional-order system is based on the formula proposed in [12] for numerical inversion of Laplace transforms. In this method the impulse response of the given fractional-order system is approximated by numerical inversion of its transfer function. The MATLAB code of this method, invlap.m, can freely be downloaded from http://www.mathworks.com/matlabcentral/fileexchange/. Most of the following examples have already been studied by author in [10].

Example 1. Consider a system with characteristic equation

$$\Delta_1(s) = (s^{\pi/2} + 1)(s^{\pi/3} + 1) \qquad (28)$$

$$= s^{5\pi/6} + s^{\pi/2} + s^{\pi/3} + 1 = 0. \qquad (29)$$

The roots of this equation can be calculated analytically, which are as the following:

$$s_{k_1} = e^{j2(2k_1+1)}, \quad k_1 \in \mathbb{Z}, \qquad (30)$$

and

$$s_{k_2} = e^{j3(2k_2+1)}, \quad k_2 \in \mathbb{Z}. \qquad (31)$$

As it is observed, the characteristic equation given in (29) has infinite many roots which are distributed on a Riemann surface with infinite number of Riemann sheets. It is concluded from (30) and (31) that (29) has four roots on the first Riemann sheet which are $e^{\pm j2}$ and $e^{\pm j3}$, and none of them are located in the right half-plane (recall that all roots whose phase angle lies in the range $[-\pi, \pi)$ belong to the first Riemann sheet).

Comparing (29) with (1) and (3) yields $\alpha_n = 5\pi/6$ (note that (29) has no delay terms). Application of (27) assuming that the lower and upper bound of integral in (27) are equal to 0 and 1000, respectively, yields $M = 2.3300 \times 10^{-4}$ which is
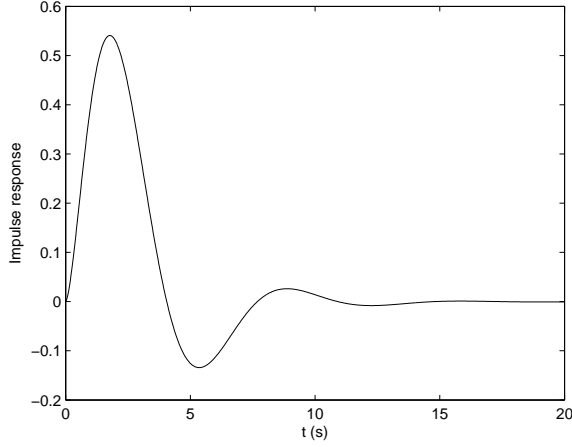
Fig. 2.   Impulse response of a system with transfer function (32).

Fig. 3.   Impulse response of a system with transfer function (34).

Fig. 4.   Impulse response of a system with transfer function (36).

consistent with the above-mentioned analytical result. Figure 2 shows the impulse response of a system with transfer function

$$H_1(s) = \frac{1}{\Delta_1(s)} = \frac{1}{s^{5\pi/6} + s^{\pi/2} + s^{\pi/3} + 1}. \qquad (32)$$

As it can be observed in this figure, the impulse response of the system is absolutely summable, as it is expected.

Example 2. Stability of a system with fractional-delay characteristic equation:

$$\Delta_2(s) = s + K(\sqrt{s} + 1)e^{-\sqrt{s}} = 0, \qquad (33)$$

is studied in [13] and it is especially shown that it is stable for $K < 21.51$ and unstable for $K > 21.51$. Application of (27) assuming that $K = 21$, $\alpha_n = 1$, and the lower and upper bound of integral are equal to 0 and 500, respectively, yields $M = 3.4227 \times 10^{-9}$, which implies the stability of system as it is expected. Figure 3 shows the impulse response of a system with transfer function

$$H_2(s) = \frac{1}{\Delta_2(s)} = \frac{1}{s + 21(\sqrt{s} + 1)e^{-\sqrt{s}}}. \qquad (34)$$

As it can be observed, the impulse response is absolutely summable, as it is expected. Repeating the above procedure with $K = 22$ yields $M = 2.0174$, which means that in this case the system has two unstable poles. This result is also consistent with the one presented in [10].

Example 3. It is shown in [14] that a system with characteristic equation

$$\Delta_3(s) = s^{1.5} - 1.5s + 4s^{0.5} + 8 - 1.5se^{-\tau s} = 0, \qquad (35)$$

is stable for the values of $\tau \in (0.99830, 1.57079)$ and unstable for other values of τ. It is also shown by author in [10] that this system has two unstable poles for $\tau = 0.99$. Application of (27) assuming $\tau = 1$ and considering the fact that here we have $\alpha_n = 1.5$ yields $M =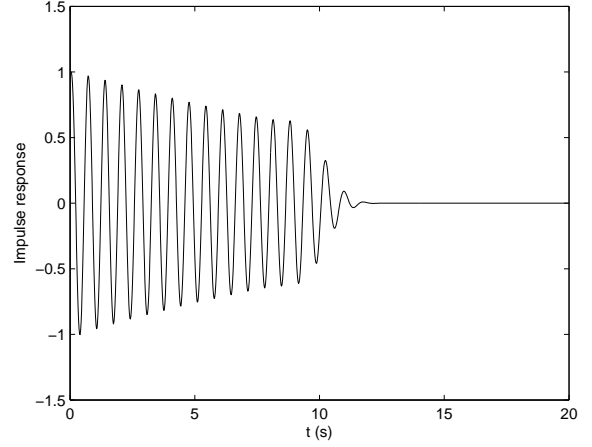 0.0082$ (the lower and upper bound of integral are considered equal to 0 and 500, respectively). As it is observed, the result obtained by using the proposed method is fairly close to zero. Figure 4 shows the impulse response of a system with transfer function:
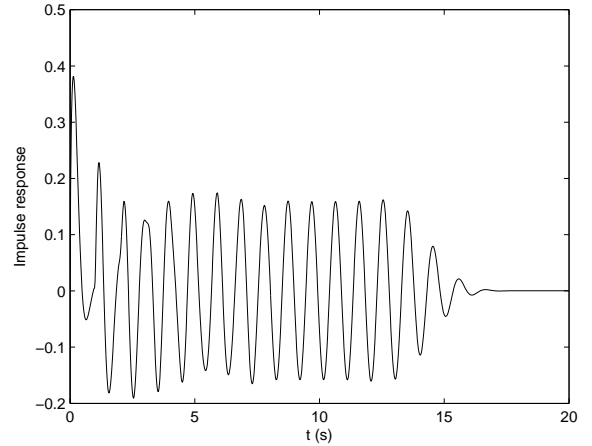
$$H_3(s) = \frac{1}{\Delta_3(s)} = \frac{1}{s^{1.5} - 1.5s + 4s^{0.5} + 8 - 1.5se^{-s}}. \qquad (36)$$

As it can be observed in this figure, the impulse response of the system is absolutely summable and consequently, the corresponding system is stable. In this example, application of (27) assuming $\tau = 0.99$ yields 1.9994 which is consistent with the result presented in [10].

Example 4. It is shown in [7] (by applying Lambert W function) that a system with the following characteristic equation

$$\Delta_4(s) = s^{5/6} + (s^{1/2} + s^{1/3})e^{-0.5s} + e^{-s} = 0, \qquad (37)$$

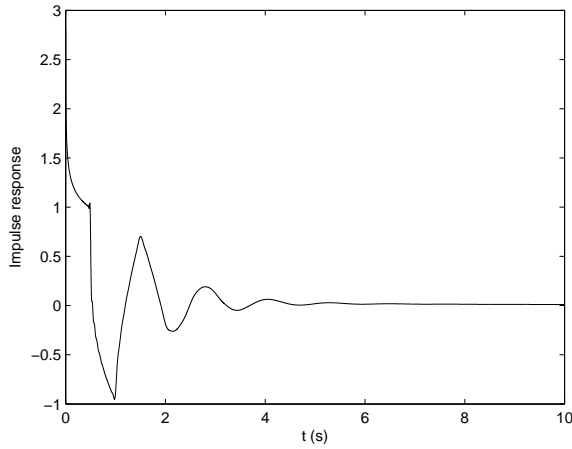is stable. Clearly, here we have $\alpha_n = 5/6$. Application of

Fig. 5. Impulse response of a system with transfer function (38).

(27) (assuming that the lower and upper bound of integral are equal to 0 and 100, respectively) leads to $M = 0.0290$, which implies the stability of system. Figure 5 shows the impulse response of a system with transfer function

$$H_4(s) = \frac{1}{\Delta_4(s)} = \frac{1}{s^{5/6} + (s^{1/2} + s^{1/3})e^{-0.5s} + e^{-s}}. \quad (38)$$

It can be observed that the impulse response is absolutely summable and consequently, the system is stable, as it is expected.

IV. CONCLUSION

An easy-to-use, effective and very general formula for stability testing of fractional-delay systems is proposed in this paper. The proposed formula can be used to determine the number of unstable poles of a system whose characteristic equation contains, in general, both fractional powers of s and exponentials of fractional powers of s.

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